“Single Channel Nonstationary Stochastic Signal Separation Using Linear Time-Varying Filters”

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What are we about to go through?

1. Introduction – the reason we’re here today
2. Signal separation using LTV filters
3. Power spectra for nonstationary stochastic processes
4. Transfer functions- “Signals and Systems” strike again...
5. Separation techniques – continuous functions
6. Separation techniques in discrete case – the picture clears up...
7. Uniform modulation – numbers and pictures at last
8. Summary and questions – for the brave who survived
Introduction

**Signal separability** - given $x(t) = d(t) + n(t)$ determine conditions on $d(t), n(t)$ to get a “good” $d^\wedge(t)$ using LTV - not the same as **Signal separation**

Temporal non-overlapping signals - can be separated by a switch (LTV) but not by LTI -
**Separability is a function of separation method**

Stochastic processes with non-overlapping power spectra - separation by bandpass filter - LTI

We have to use an a priori knowledge of the signal structure in a single channel problem

Time - switch, frequency - bandpass. If we could find a domain for the signals to be disjoint in, from the signals themselves...
Signal separation using LTV filters

“Perfect” separation of stochastic signals - \( \sigma^2(t) = E\left\{ \hat{d}(t) - d(t) \right\}^2 \right\} = 0 \quad \forall t \in T \)

Second order cost function -> second order statistics (ACF) \( R_{xx}(t, \tau) \)

Separation - \( \hat{d}(t) = \int_{T} h(t, \alpha)x(\alpha)d\alpha \), \( h(t, \alpha) \) minimizes \( \sigma^2(t) \)

Your cost function – your rules. MSE leads to nonstationary WHF

\[
R_{dx}(t, \tau) = \int_{T} h(t, \alpha) R_{xx}(\alpha, \tau) d\alpha
\]

\[
MSE = R_{dd}(t, t) - \int_{T} h(t, \alpha) R_{dx}(t, \alpha)d\alpha = 0 \text{ if P.S}
\]
Power Spectra for Nonstationary Stochastic Processes

Deterministic signals

\[ x(t), \ t \in T \Rightarrow X(\lambda), \ \lambda \in \Lambda \ \Rightarrow X(\lambda) = \int_{T} x(t) K(t, \lambda) dt, \ \forall \lambda \in \Lambda \]

\[ x(t) = \int_{\lambda} X(\lambda) k(\lambda, t) d\lambda \ \forall \ t \in T \]

The transformation is isomorphic – uniqueness of representation

\[ K(t, \lambda), k(\lambda, t) \text{ must satisfy } : \delta(t - \tau) = \int_{\Lambda} K(t, \lambda)k(\lambda, \tau)d\lambda \ \forall (t, \tau) \in T^2 \]

\[ \delta(\lambda - \hat{\lambda}) = \int_{T} k(\lambda, t)K(t, \hat{\lambda})dt \ \forall (\lambda, \hat{\lambda}) \in \Lambda^2 \]

\( \delta \) - Dirac \( \delta \) function

If they do -> \( k(\lambda, t), K(t, \lambda) \) are a Transform pair over \( T \times \Lambda \)
Power Spectra for Nonstationary Stochastic Processes

Stochastic Spectral Transforms

Same as for deterministic signals but the decomposition is for a particular realization

\[ X(\lambda) = \int_{T} x(t) K(t, \lambda) dt, \forall \lambda \in \Lambda \]

\[ \hat{x}(t) = \int_{\lambda} X(\lambda) k(\lambda, t) d\lambda \quad \forall t \in T \]

\[ \hat{x}(t) \text{ equals } x(t) \text{ in the MS sense} \quad \mathbb{E}\{\left| \hat{x}(t) - x(t) \right|^2\} = 0 \]

From these, we can determine the 2nd order statistics
Power Spectra for Nonstationary Stochastic Processes

Generalized Power Spectrum - GPS

\[ \rho_{xx}(\lambda, \hat{\lambda}) = E\{X(\lambda)X^*(\lambda)\} \]

Using the spectral transforms we get:

\[ \rho_{xx}(\lambda, \hat{\lambda}) = \int T^{2} R_{xx}(t, \tau)K(t, \lambda)K^*(\tau, \hat{\lambda})dt d\tau \]

\[ R_{xx}(t, \tau) = \int \int \rho_{xx}(\lambda, \hat{\lambda})k(\lambda, t)k^*(\hat{\lambda}, \tau)d\lambda d\hat{\lambda} \]

Generalized Cross Power Spectrum - GCPS

Introducing another stochastic signal y(t)

\[ \rho_{yx}(\lambda, \hat{\lambda}) = \int T^{2} \rho_{yx}(t, \tau)K(t, \lambda)K^*(\tau, \hat{\lambda})dt d\tau \forall (\lambda, \hat{\lambda}) \in \Lambda^{2}_{0} = \Lambda_{0} \times \Lambda_{0} \]

\[ R_{yx}(t, \tau) = \int \int \rho_{yx}(\lambda, \hat{\lambda})k(\lambda, t)k^*(\hat{\lambda}, \tau)d\lambda d\hat{\lambda} \forall (t, \tau) \in T^{2} \]

\[ \Lambda_{0} - \text{the region of } \Lambda \text{ over which the spectral components of } x, y \text{ overlap} \]
Transfer Functions

Transfer function of a linear system gives us the relationship between its input and output via superposition integral

Generally, \( y(t) = \int_T h(t, \tau)x(\tau)d\tau \) for LTV, reducing to convolution for LTI

If \( y(t) = L\{x(t)\} \), \( L \)- linear operator filtering, then

\[
h(t, \tau) = \iiint_{\Lambda^2} H(\lambda, \hat{\lambda})k(\lambda, t)K(\tau, \hat{\lambda})d\lambda d\hat{\lambda} \quad \forall (t, \tau) \in T^2
\]

\[
H(\lambda, \hat{\lambda}) = \iiint_{T^2} h(t, \tau)K(t, \lambda)k(\hat{\lambda}, \tau)dtd\tau \quad \forall (\lambda, \hat{\lambda}) \in \Lambda^2
\]

\( H(\lambda, \hat{\lambda}) \) is called GBTF of \( L \)

Spectral Convolution:

\[
Y(\lambda) = \int_{\Lambda} H(\lambda, \hat{\lambda})X(\hat{\lambda})d\hat{\lambda}
\]
Separation Techniques

Ideal Filter

Ideal Bandpass Filter – passes all frequency components falling in its passband, rejecting all others

Generalized Ideal Filter does the same with generalized spectral components on an arbitrary domain

$L$ is a GIF over $\Lambda$ for input space $X$ if for $y = L\{x\}$ $Y(\lambda)$ has the form of

$$Y(\lambda) = \begin{cases} 
0 & \lambda \notin \Lambda_H, \Lambda_H \in \Lambda \\
X(\lambda) & \lambda \in \Lambda_H
\end{cases}$$

$L$ is a GIF iff $K(t, \lambda), k(\lambda, t)$ exist satisfying transform pair conditions such that

$$L\{k(\lambda, t)\} = \begin{cases} 
0 & \lambda \notin \Lambda_H, \Lambda_H \in \Lambda \\
k(\lambda, t) & \lambda \in \Lambda_H
\end{cases}$$

Impulse response: for an ideal filter over $\Lambda$ with impulse response $h(t, \tau)$

$$h(t, \tau) = \int_{\Lambda_H} k(\lambda, t) K(\tau, \lambda) d\lambda \ \forall (t, \tau) \in T^2$$
Separation Techniques

We have to solve \( R_{dx}(t, \tau) = \int_{T} h(t, \alpha)R_{xx}(\alpha, \tau)d\alpha \) to get WHF.

Assuming the following decomposition:

\[
\begin{align*}
R_{dd}(t, \tau) &= \int_{\Lambda^2_d} \rho_{dd}(\lambda, \hat{\lambda})k(\lambda, t)k^*(\hat{\lambda}, \tau)d\lambda d\hat{\lambda} \\
R_{nn}(t, \tau) &= \int_{\Lambda^2_n} \rho_{nn}(\lambda, \hat{\lambda})k(\lambda, t)k^*(\hat{\lambda}, \tau)d\lambda d\hat{\lambda} \\
R_{dn}(t, \tau) &= \int_{\Lambda^2_{0}} \rho_{dn}(\lambda, \hat{\lambda})k(\lambda, t)k^*(\hat{\lambda}, \tau)d\lambda d\hat{\lambda}
\end{align*}
\]

\( \hat{\Lambda}_d \equiv \Lambda_d \oplus \Lambda_0, \hat{\Lambda}_n \equiv \Lambda_n \oplus \Lambda_0, \Lambda_d \cap \Lambda_n = \{\phi\} \)

Sufficient solution is the GBTF:

\[
\begin{align*}
H(\lambda, \hat{\lambda}) &= \begin{cases} \\
\delta(\lambda - \hat{\lambda}), & (\lambda, \hat{\lambda}) \in \Lambda^2_d \\
H_0(\lambda - \hat{\lambda}), & (\lambda, \hat{\lambda}) \in \Lambda^2_0 \\
0 & \text{else}
\end{cases}
\end{align*}
\]

\( H_0(\lambda, \hat{\lambda}) \) is the solution of \( \rho_{dx}(\lambda, \hat{\lambda}) = \int_{\Lambda^2_0} H_0(\lambda, \overline{\lambda})\rho_{dx}(\overline{\lambda}, \hat{\lambda})d\overline{\lambda} \quad \forall(\lambda, \hat{\lambda}) \in \Lambda^2_0 \)

\( H_0(\lambda, \hat{\lambda}) = 0 \quad \forall(\lambda, \hat{\lambda}) \in \Lambda^2_0 \)
Separation Techniques

The filter \( h(t, \tau) \) is:

\[
\hat{h}(t) = \int_{\Lambda_d} k(\lambda, t) K(\tau, \lambda) d\lambda + \int_{\Lambda_0^2} H_0(\lambda, \hat{\lambda}) k(\lambda, t) K(\tau, \hat{\lambda}) d\lambda d\hat{\lambda}
\]

**MSE:**

\[
\sigma^2(t) = \int_{\Lambda_0^2} \rho_{\sigma\sigma}(\lambda, \hat{\lambda}) k(\lambda, t) k^*(\hat{\lambda}, t) d\lambda d\hat{\lambda}
\]

1 corresponds to an ideal filter passing the generalized frequency range \( \Lambda_d \) and depends solemnly on \( \Lambda_d \)

\( \Lambda_0 = \{ \Phi \} \rightarrow \text{MSE} = 0 \)

The WHF \( h(t, \tau) \) for perfect separation is an ideal filter if the signals are disjoint in the generalized spectrum components.

Independence of signal values means separation of all processes disjoint in filter’s domain.
Separation Techniques

Prior derivations are based upon existence of the $\lambda$ domain, no recipe of choosing it yet.

Common domain is found by “concatenating” the basis functions for individual signals.

If $d(t)$, $n(t)$ are band limited in their respective domains, concatenation is possible.

Fig. 3. Concatenating power spectra.
Separation Techniques
Separating Modulated Signals

\[
d(t) = \int_{-T}^{T} h_d(t, \tau) a(\tau) d\tau \\
n(t) = \int_{-T}^{T} h_n(t, \tau) b(\tau) d\tau
\]
\[
t \in T \quad h_d, h_n \text{ known, deterministic, } a(t), b(t) \text{ band limited to } \pm \omega_c
\]
a(t), b(t) band limited, therefore

\[
a(t) = \int_{-\omega_c}^{\omega_c} A(\omega)e^{i\omega t} d\omega \\
b(t) = \int_{-\omega_c}^{\omega_c} B(\omega)e^{i\omega t} d\omega
\]

With appropriate transformations

\[
d(t) = \int_{0}^{2\omega_c} D(\lambda)k(\lambda, t)d\lambda \\
n(t) = \int_{-2\omega_c}^{0} N(\lambda)k(\lambda, t)d\omega
\]

\[
k(\lambda, t) = \left\{ \begin{array}{ll}
\int_{-T}^{T} h_d(t, \tau)e^{i(\lambda-\omega_c)\tau} d\tau & \lambda \in [0, 2\omega_c) \\
\int_{-T}^{T} h_n(t, \tau)e^{i(\lambda+\omega_c)\tau} d\tau & \lambda \in [-2\omega_c, 0) \end{array} \right.
\]

\[
k_v(\lambda, t) \lambda \not\in (-2\omega_c, 2\omega_c)
\]

If d, n are separable, k, K satisfy transform pair properties, so constraints on both filters h_d, h_n can be derived.
Separation Techniques

Discrete Case

In the discrete case, transform can be viewed as change in basis vectors.

If \( \{ x(n), n \in N \in Z \} \) is discrete time signal, \( \{ X(p), p \in P \in Z \} \) its representation, then

\[
X(p) = \sum_{n \in N} x(n)K(n, p), \quad x(n) = \sum_{p \in P} X(p)k(p, n)
\]

\( K, k \) satisfy

\[
\sum_{p \in P} K(n, p)k(p, \hat{n}) = \delta(n - \hat{n})(n, \hat{n}) \in (N \times N)
\]

\[
\sum_{p \in N} k(p, n)K(n, \hat{p}) = \delta(p - \hat{p})(p, \hat{p}) \in (P \times P)
\]

\( \delta \) - Kroenecker's delta

The sums above can be expressed as \( X = K^T x \), \( x = k^T X \)

\( kK = Kk = I_N \rightarrow \text{Basis kernels must have full rank} \)
Separation Techniques

Discrete Case

Concatenation of discrete spectra is done similarly to the continuous case.

The concatenated kernel must have an “inverse” for the desired domain to exist -> matrix inversion in the discrete case!

In discrete modulation, \( k^T = [H_d \hat{k}_d^T \mid H_n \hat{k}_n^T \mid \hat{k}_v^T] \) where \( k_v \) are unused basis vectors.

\[
\begin{align*}
[\hat{k}_d]_{pn} &= W_n^{p-q_c}, \quad [\hat{k}_n]_{pn} = W_n^{p+q_c}, \quad [\hat{k}_v]_{pn} = k_v(p, n) \\
W_n^{np} &= (1/N)e^{jnp(2\pi/N)}, \quad [H_{[\theta]}]_{pn} = h_{[\theta]}(n, \hat{n}) \quad [\Theta] = \{d, n\}
\end{align*}
\]

Constraints on \( H_d, H_n \) once again must be found for \( k \) to be full ranked.
**Separation Techniques**

**Discrete Case**

If \( k \) is full rank, signals are separable

An ideal filter matrix \( A \) is built using \( k^T \) previously defined, \( \hat{d} = Ax \)

If noise adds up to the input, it will appear at the output as well, affected by the spectral properties and noise gain of \( A \)

\[
E\left( \sum_{i=1}^{N} y_i^2 \right) = \frac{1}{N} \text{trace}[AA^T] = \frac{1}{N} \| A \|_F
\]

For white noise, \( \eta_{WN} \equiv \frac{1}{N} \sum_{i=1}^{N} w_i^2 \)
Examples - Uniform Modulation

\( y(t) = x(t)c(t) \), \( x(t) \) stochastic process, \( c(t) \) known deterministic process

Accordingly, \( d(t) = h_d(t)a(t) \), \( n(t) = h_n(t)b(t) \)

Kernel expressions reduce to:

\[
 k(\lambda, t) = \begin{cases} 
 h_d(t) \cdot e^{j(\lambda - \omega_c)t} & \lambda \in [0, 2\omega_c) \\
 h_n(t) \cdot e^{j(\lambda + \omega_c)t} & \lambda \in (-2\omega_c, 0] 
\end{cases}
\]

\[ k_v(\lambda, t) = e^{j\lambda t} \quad \forall \lambda \notin (-2\omega_c, 2\omega_c) \]

serves as basis extension

If \( k^T \) has full rank, we build \( A \) and perform the separation
Example 1 - Chirp modulated signals

\[
h_d(t) = \cos [2\pi f_i \frac{t}{f_s} \left(1 + \frac{1}{\tau_d} \frac{f_f - f_i}{f_i} t\right)] \quad t \in \{0, \ldots, T - 1\}
\]

\[
h_n(t) = \cos [2\pi f'_i \frac{t}{f_s} \left(1 + \frac{1}{\tau_n} \frac{f'_f - f'_i}{f'_i} t\right)]
\]

a(t), b(t) are band limited to \( f_c \), with \( f_s \geq 4f_c \)

In this example,

\( f_i = f'_f = 4250Hz, f_f = f'_i = 5750Hz, f_s = 22.050KHz, \tau_n = \tau_d = 70msec \)

\( T = 1000 \) (45msec of data) \( f_c = 5KHz \) (\( p_c = 227 \) freq.bins)

Kernel matrix has full rank -> signals are separable
Example 1 - Chirp modulated signals

Fig. 4. Fourier spectra of $h_d(t), h_m(t); f_c = 22.050 \text{ kHz}$.

Impulse response for recovering $d(t)$.

Fig. 5. Response of the ideal filter that recovers $d(t)$ for the chirp-modulated case.

Fig. 6. Unknown stochastic modulating signals $a(t), b(t)$. The sampling frequency $f_s = 22.050 \text{ kHz}$, and they are bandlimited to 5 kHz, as emphasized in (b). (a) Time series for $a(t)$ and $b(t)$. (b) Fourier transform of $d(t)$ and $u(t)$. 

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Example 1 - Chirp modulated signals

Fig. 7. Desired $d(t)$ and noise $n(t)$ signals resulting from multiplying the signals in Fig. 6(a) by the chirp signals of Fig. 6. The Fourier transforms of $d(t)$ and $n(t)$ are overlapping and would, therefore, conventionally be considered inseparable. (a) Time series for $d(t)$ and $n(t)$. (b) Fourier transforms of $d(t)$ and $n(t)$.

Fig. 8. (Top) Fourier transform of the observed signal $x(t) = d(t) + n(t)$, for the resulting signals in Fig. 7(a). (Bottom) Generalized transform of $x(t)$.

Fig. 9. Generalized spectra of $d(t)$ and $n(t)$, obtained by bandpass filtering the generalized spectrum of $x(t)$, as shown in Fig. 8 with passbands $\{0, 2f_r, -1\}$ and $\{2f_r, 4f_r, -1\}$, respectively.
Example 2 - Gaussian Chirp modulated signals

\[ h_d(t) = \cos\left[ 2\pi f_i t \left( 1 + \frac{1}{\tau_d} \frac{f_f - f_i}{f_i} t \right) \right] \cdot \exp\left\{ -\frac{(t - \hat{\tau}_d)^2}{\sigma_d^2} \right\} \]

\[ h_n(t) = \cos\left[ 2\pi f_i' t \left( 1 + \frac{1}{\tau_n} \frac{f_f' - f_i'}{f_i'} t \right) \right] \cdot \exp\left\{ -\frac{(t - \hat{\tau}_n)^2}{\sigma_n^2} \right\} \]

\[ \hat{\tau}_n = \hat{\tau}_d = T / 2 \]

\[ \sigma_d = \sigma_n = 1/(20\sqrt{5}) \]

Fig. 11. Fourier spectra of \( h_d(t), h_n(t); f_s = 22.050 \text{ kHz} \).

Fig. 12. Response of the ideal filter that recovers \( d(t) \) for the Gaussian chirp-modulated case.
Future Challenges

- Accurate parameter estimation of the signals
- Compensation of errors in estimating $h_d, h_n$
- Robustness improvement
- Continuous case research
Summary

- LTV separation technique was defined
- Nonstationary Wiener-Hopf filter was introduced
- GPS and GCPS were defined
- Transfer function of a LTV system (GBTF) was introduced
- The ideal filter notion and its impulse response were derived
- WHF sufficient solution was given
- The concatenation separation technique was introduced
- Separability constraints of filter modulated signals were shown
- The discrete case of modulation was treated
- Two classes of uniformly modulated signals were successfully separated
Questions Time...